Sheet 4

P10

a) In the focus: $I = I_0 \, e^{-\frac{2r^2}{w_0^2}}$

$$\Rightarrow \text{ Power: } P = \int_0^\infty \int_0^{2\pi} I(r) \cdot r\,dr\,d\varphi$$

$$= I_0 \cdot 2\pi \int_0^\infty r\, e^{-\frac{2r^2}{w_0^2}}\, dr$$

$$= I_0 \cdot 2\pi \int_0^\infty -\frac{1}{4} w_0^2 \frac{\partial}{\partial r} e^{-\frac{2r^2}{w_0^2}}\, dr$$

$$= \frac{\pi w_0^2}{2} I_0 \cdot \underbrace{\left[e^{-\frac{2r^2}{w_0^2}} \right]_0^{-\infty}}_{1}$$

$$= \frac{\pi w_0^2}{2} I_0$$

$$\Rightarrow \quad \underline{I_0 = \frac{2P}{\pi w_0^2}} \quad \text{qed.}$$

$\boxed{P10}$

b)

$$\frac{1}{q} = \frac{1}{R} - i\,\frac{\lambda}{\pi w^2}$$

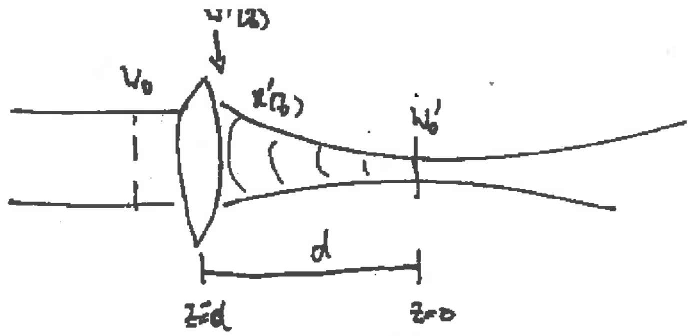
$$\Rightarrow \quad \frac{1}{R_2} - i\,\frac{\lambda}{\pi w_2^2} = \frac{1}{R_1} - i\,\frac{\lambda}{\pi w_1^2} - \frac{1}{f} \qquad (1)$$

$$\mathrm{Im}(1) \;\hat{=}\; \frac{1}{\pi w_2^2} = \frac{1}{\pi w_1^2} \;\Rightarrow\; w_2 = w_1 \quad \text{as} \quad w_i \geq 0$$

$$\mathrm{Re}(1) \;\hat{=}\; \frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \qquad \text{qed.}$$

P10

c)



1.) $$w'(d) = W_0 \overset{!}{=} W_0' \sqrt{1 + \frac{d^2}{z_R'^2}}$$

$$\Rightarrow \frac{W_0^2}{W_0'^2} = 1 + \frac{d^2}{z_R'^2}$$

$$\Leftrightarrow \frac{d^2}{z_R'^2} = \frac{W_0^2 - W_0'^2}{W_0'^2}$$

$$\Leftrightarrow \underline{z_R'^2 = \frac{d^2 W_0'^2}{W_0^2 - W_0'^2}} \quad (I)$$

2.) $$\frac{1}{R'(d)} = \frac{1}{\infty} - \frac{1}{f} = -\frac{1}{f}$$

$$\Rightarrow R'(d) = -f \overset{!}{=} -d + \frac{z_R'^2}{-d}$$

$$\Rightarrow d^2 + z_R'^2 = df$$

$$\Leftrightarrow \underline{z_R'^2 = df - d^2} \quad (II)$$

(II) in (I) $$\Rightarrow \frac{d^2 W_0'^2}{W_0^2 - W_0'^2} = df - d^2$$

$$\Leftrightarrow \frac{W_0'^2}{W_0^2 - W_0'^2} = \frac{f}{d} - 1 \quad \Rightarrow \underline{d = f\left(1 - \frac{W_0'^2}{W_0^2}\right)} \quad (III)$$

or $$W_0'^2 = W_0^2 \cdot \frac{f-d}{f} \quad (III')$$

3.) $$z_R' = \frac{\pi w_0'^2}{\lambda} \overset{?}{=} \sqrt{df - d^2}$$

$$\Rightarrow \quad w_0'^4 = \frac{\lambda^2}{\pi^2} \cdot \left(df - d^2\right) \quad \text{(IV)}$$

$[\text{III}']^2 = (\text{IV})$:

$$\frac{w_0^4 \, (d-f)^2}{f^2} = \frac{\lambda^2}{\pi^2} \left(df - d^2\right)$$

$$\Rightarrow \quad \frac{w_0^4 \, (d-f)}{f^2} = -\frac{\lambda^2}{\pi^2} d$$

$$\Rightarrow \quad f - d = \frac{\lambda^2}{\pi^2} \, \frac{d f^2}{w_0^4}$$

$$\Rightarrow \quad d \left(\frac{\lambda^2 f^2}{\pi^2 w_0^4} + 1 \right) = f$$

$$\Rightarrow \quad \boxed{d = \frac{f}{1 + \frac{\lambda^2 f^2}{\pi^2 w_0^4}}} \qquad \text{q.e.d.}$$

from (III') we obtain

$$w_0'^2 = w_0^2 \cdot \frac{f - \frac{f}{1 + \lambda^2 f^2 / \pi^2 w_0^4}}{f}$$

$$= w_0^2 \cdot \frac{\frac{\lambda^2 f^2}{\pi^2 w_0^4}}{1 + \frac{\lambda^2 f^2}{\pi^2 w_0^4}} = w_0^2 \cdot \frac{1}{\frac{\pi^2 w_0^4}{\lambda^2 f^2} + 1} \Rightarrow w_0' = \frac{w_0}{\sqrt{1 + \frac{\pi^2 w_0^4}{\lambda^2 f^2}}} \quad \text{q.e.d.}$$

P11

a) $$\Theta \cdot w_0 = \frac{M^2 \lambda}{\pi}$$

$$\Rightarrow \quad M^2 = \frac{\pi}{\lambda} \cdot \Theta \cdot w_0 = 2.3$$

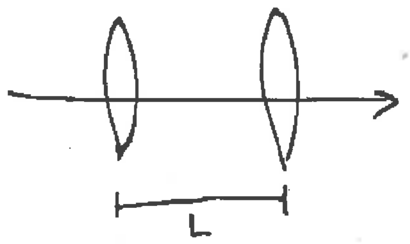
b) $$\lambda' = M^2 \lambda = 2.37 \mu m$$

c) $$w_0' = 25 \mu m$$

$$\Rightarrow \quad \Theta' = \frac{M^2 \lambda}{\pi w_0'} = \frac{2.3 \cdot 1.029 \mu}{\pi \cdot 25 \mu} = 30 \text{ mrad}$$

P12)

a)



$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

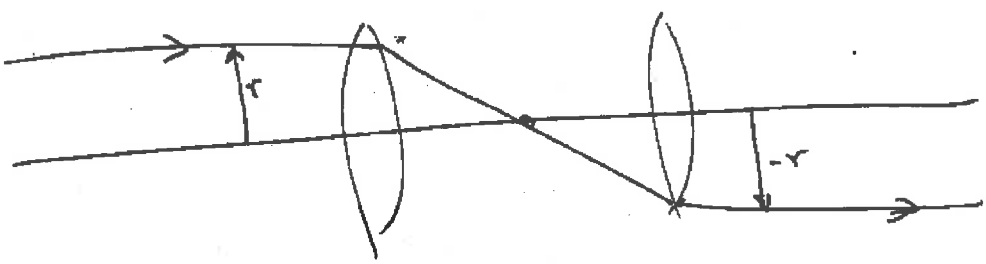
$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1-\frac{1}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{2}{f}+\frac{L}{f^2} & 1-\frac{L}{f} \end{pmatrix}$$

For $L = 2f$ we find $M = \begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix}$. A collimated ray $\begin{pmatrix} r \\ 0 \end{pmatrix}$ is thus transformed into the ray

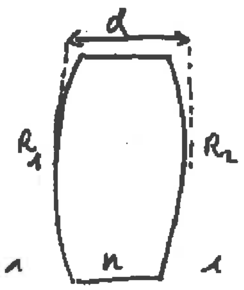
$$\vec{r}' = M\begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} -r \\ 0 \end{pmatrix}$$

which corresponds to an inverted beam with respect to the optical axis.



P12

b)



$$\Rightarrow \quad M_L = \begin{pmatrix} 1 & 0 \\ \frac{n-1}{-R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-n}{R_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1-n}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 + \frac{1-n}{n} \cdot \frac{d}{R_2} & \frac{d}{n} \\ \frac{1-n}{R_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1-n}{n} \frac{d}{R_2} & \frac{d}{n} \\ (1-n)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{(1-n)^2}{n} \frac{d}{R_1 R_2} & 1 + \frac{1-n}{n} \frac{d}{R_1} \end{pmatrix}$$

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c) For $d \to 0$ we get

$$M_L \to \begin{pmatrix} 1 & 0 \\ (1-n)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} .$$

$\Rightarrow \quad \frac{1}{f} = (n-1)\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$ lens maker formula. qed.